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# LIMIT ANALYSIS OF STIFFENED AXISYMMETRIC SHELLS*

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**Abstract**—Accurate limit ratios have been obtained for stiffened axisymmetric shells. Depending on whether the interaction at the points of intersection is taken into account or disregarded, shells are classified into those of rib-type and waffle-type. In particular cases the ratios obtained may be reduced to limit ratios for shells with ribs running in one direction, for bimetallic shells and uniform isotropic ones. Approximate limit ratios are suggested and a brief survey is given of results obtained in the analysis of other types of rib-type shells. Linearization of limit ratios by replacing the discussed shells by layer models is considered. Application of the limit ratios obtained is illustrated by a problem of the carrying capacity of console stiffened cylindrical shells loaded with internal hydrostatic pressure. Formulas are obtained which allow one to estimate the efficiency of using one or another type of strengthening.

## INTRODUCTION

A LARGE number of research works have been devoted to plastic analysis of plates and shells made from ideal rigid–plastic isotropic materials and obeying the Tresca yield condition (some of these works are summarized in [1]). The plastic analysis of the designs fabricated of physically anisotropic material obeying the piecewise linear yield conditions was carried out in [2–4.] However while in the elastic strain region the constructive and physical anisotropies of the strained body are indistinguishable, for they can be expressed by similar equations, in the plastic strain region the defining equations essentially depend on the type of anisotropy. Therefore each type of anisotropy must be analysed separately when passing beyond the elasticity limits.

The plastic analysis of the ribbed shells was carried out in [5–9], that of concrete shells [7, 10] and that of multilayered axisymmetric reinforced shells in [11]. In [5–9] the analysis of plastic states of ribbed shells was restricted either by the case of cylindrical shells or by some simplifying models.

The present paper presents an accurate analysis of states of the limit equilibrium of ribbed shells when applying the Tresca yield condition for materials and the associated flow law (interior hexagon, Fig. 1). For the single-layer unsymmetrically stiffened shells [5] consisting of a shell layer and a stiffened layer, which is unsymmetric with respect to the middle shell surface, accurate plastic relations as well as corresponding inequalities and the plastic flow law have been obtained. The latter can be expressed by tensions, moments, strain rates and curvature rates of the middle shell surface. Then stiffened shells are divided into ribbed shells when interaction in the assemblies of the rib intersection of the stiffening layer are not taken into account (the model of the stiffening layer is shown in Fig. 2) and the waffle-type (shown in Fig. 3) when this interaction is taken into account.

In particular cases the derived plastic relations are transformed to corresponding relations for shells stiffened by single-direction ribs, for bimetal and smooth homogeneous

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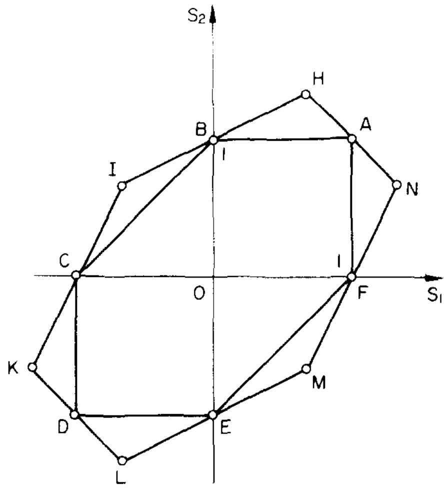


FIG. 1.

isotropic shells [12]. A transition to cylindrical shells has been effected. In particular, accurate plastic relations were obtained for the non-symmetric stiffened by ribs [5] and waffle-type [9] cylindrical shells. Approximate plastic relations are proposed for axisymmetrical stiffened waffle-type and ribbed shells.

The solution of the problem of carrying capacity of cantilever stiffened cylindric shell loaded internally by uniformly distributed pressure.

## 1. YIELD CONDITION FOR THE STIFFENING LAYER

Assuming that the material of all the elements composing the stiffened shell is an ideal rigid–plastic ones obeying the Tresca yield condition and associated flow law (Fig. 1) [1], we first determine the yield condition for the stiffening layer shown in Fig. 3 under two axial extension-compression forces.

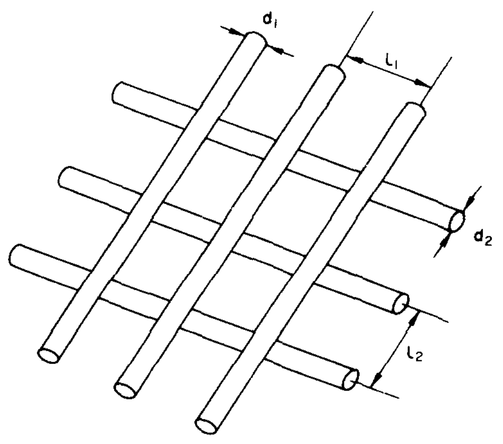


FIG. 2.

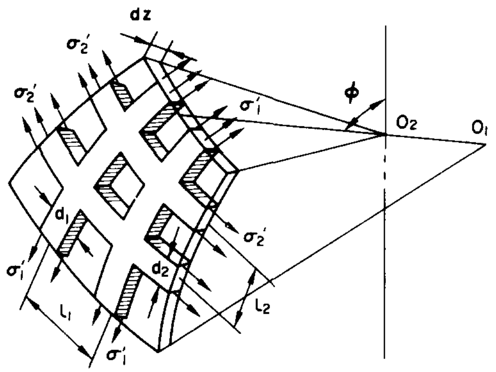


FIG. 3.

For a general case we admit that the yield stresses of all the elements of the stiffening layer are different under extension and compression. Let the element of the stiffening layer be under stresses $\sigma_i'$ in the $i = 1, 2$ direction.

The carrying capacity of the stiffening layer will be exhausted if any of its strain rates $\varepsilon_1$ or $\varepsilon_2$ is not zero.

The strain rates of the stiffening layer are expressed by the strain rates of elements composing it as follows:

$$\varepsilon_1 = \omega_2\varepsilon_1^a + (1-\omega_2)\varepsilon_1^r, \qquad \varepsilon_2 = \omega_1\varepsilon_2^a + (1-\omega_1)\varepsilon_2^r. \tag{1.1}$$

Here $\varepsilon_i^a$ denote the strain rates of assembly in the direction $i$, $\varepsilon_i^r$ denote the strain rates of the ribs of $i$-th direction, $\omega_i = d_i/l_i$, all other notations are clear from Figs. 2 and 3.

We consider possible cases of exhausted carrying capacity. Let the stresses $\sigma_i'$—be of the same sign. Then the capability of the stiffening layer will be exhausted, if

$$s_i' = \pm s_{0i}^{\pm} \quad \text{for} \quad s_{0i}^{\pm} \le s_0^{\pm} \tag{1.2}$$

$$s_i' = \pm s_0^{\pm} \quad \text{for} \quad s_{0i}^{\pm} \ge s_0^{\pm}\,. \tag{1.3}$$

Here $s_i' = \sigma_i'/\sigma_0$, $\sigma_0$ denotes the yield stresses of the facing material, $s_{0i}^{\pm}$ the dimensionless yield stresses of rib material under extension (+) and compression (−), $s_0^{\pm}$ denotes the dimensionless yield stresses of the assembly material under extension and compression (for stiffening layer shown in Fig. 3).

Then in accordance with the flow law for the elements of the stiffening layer we have

$$\varepsilon_j^r = \varepsilon_j^a = 0, \qquad \pm\varepsilon_i^r > 0, \qquad \pm\varepsilon_i^a \ge 0 \qquad (i, j = 1, 2\,; i \ne j)$$

for the first case and

$$\varepsilon_j^r = \varepsilon_j^a = 0,\ \pm\varepsilon_i^r \ge 0, \qquad \pm\varepsilon_i^a > 0$$

for the second. Hence, because of (1.1), for the two cases we have:

$$\varepsilon_j = 0, \qquad \pm\varepsilon_i > 0$$

If the stresses $\sigma_1'$ and $\sigma_2'$ are of opposite signs then the carrying capacity of the stiffening layer will be exhausted after observing the equalities

$$\frac{s_2'}{s_0^{\pm}} - \frac{s_1'}{s_0^{\mp}} = \pm 1 \quad \text{if} \quad \frac{s_{02}^{\pm}}{s_0^{\pm}} + \frac{s_{01}^{\mp}}{s_0^{\mp}} \le 1 \tag{1.4}$$

or (1.2), if

$$\frac{s_{02}^{\pm}}{s_0^{\pm}}+\frac{s_{01}^{\mp}}{s_0^{\mp}} \geq 1 \tag{1.5}$$

In this case for equation (1.4) we have:

$$\varepsilon_1^r = \varepsilon_2^r = 0, \qquad s_0^{\pm}\varepsilon_2^a + s_0^{\mp}\varepsilon_1^a = 0$$

or in accordance with (1.1)

$$s_0^{\pm}\omega_2\varepsilon_2 + s_0^{\mp}\omega_1\varepsilon_1 = 0. \tag{1.6}$$

Equations (1.2)–(1.4) in the plane $s_1', s_2'$ define the limit hexagon $A_1B_1C_1D_1E_1F_1$ or tetragon $A_1H_1D_1G_1$ (Fig. 4). If inequality (1.5) is valid only for the super or subscripts, then the plastic analysis of the stiffening layer corresponds to $A_1H_1D_1E_1F_1$ or $A_1B_1C_1D_1G_1$ pentagon.

For all the sides of these polygons, except $B_1C_1$, $E_1F_1$ the flow law corresponds to the condition of orthogonality of the strain rate vector $\vec{\varepsilon}$ with components $\varepsilon_1, \varepsilon_2$. As evident from (1.6) at the sides $B_1C_1$ and $E_1F_1$ the condition of orthogonality is violated. However, if we introduce instead of this stiffening layer and its equivalent in the sense of carrying capacity of uniform orthotropic material with the yield stress in the $i$-th direction equal to $\omega_i s_{0i}^{\pm}$, then in the stress plane $s_i'' = \omega_i s_i'$ we get a modified hexa-pentagon or tetragon for which the condition of orthogonality of the strain rate vector $\vec{\varepsilon}$ $(\varepsilon_1, \varepsilon_2)$ will be fulfilled for all sides.

Without going into details let us remark on the form of the plastic polygons for the constructive orthotropic layer. It is believed [4] that for such a layer (in particular that reinforced by strong filaments) at a suitable choice of characteristics as a yield polygon one can make use of hexagon of the same type for physically anisotropic material. In fact, the form of the yield polygon and even the number of its sides essentially depend on the type of the constructive anisotropy and properties of materials composing the anisotropic layer. Thus if the material of all the elements composing the layer shown in Fig. 3 obeys the yield condition of maximal reduced stress (external hexagon Fig. 1) [13] or it is physically orthotropic obeying the yield condition in the form of a hexagon derived in [4], then depending on the quantity of the yield stress of elements composing this layer, the yield polygon for the stiffening layer may contain 4–10 sides. For the layer of material obeying the Tresca yield condition stiffened by filaments we shall have either a hexagon of the form given in Fig. 4, or an octagon [11].

Below the stiffened shells, for which the stiffening layer (Fig. 3) obeys the yield condition in the form of pentagon or hexagon shown in Fig. 4, we call waffle-type shells, in which the stiffening layer obeys the yield condition in the form of a tetragon shown in Fig. 4, we call ribbed. The model of the stiffening layer for such a shell is shown in Fig. 2.

Below we denote the coordinates of the point $A_1$ by $\lambda_i^+$, the abscissa of the point $B_1$ by $(-a_1^-)$ and the ordinate of the point $C_1$ by $a_2^+$. Analogously we introduce the notation for other points. This provides a possibility at $\lambda_i^{\pm} \to s_{0i}^{\pm}$, $a_1^{\mp} \to \pm\lambda_1^{\pm}$, $a_2^{\pm} \to \pm\lambda_2^{\pm}$ of passing from hexagon $A_1B_1C_1D_1E_1F_1$ shown in Fig. 4 to tetragon $A_1H_1D_1G_1$ and obtaining the plastic relations valid for both waffle-type shells and ribbed shells.

## 2. LIMIT RELATIONS FOR THE STIFFENED AXISYMMETRIC SHELLS

By stiffened shells we mean shells consisting of a facing layer and a stiffening layer. It is anticipated that the ribs of the stiffening layer are placed in the direction of the main

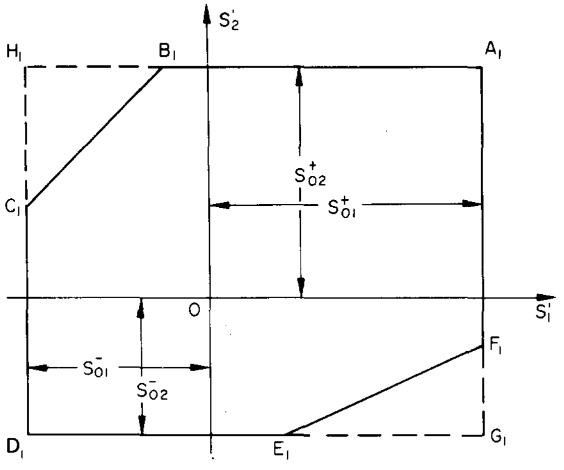


FIG. 4.

shell curvatures. This happens so often that introduction of the averaged characteristics is quite reasonable.

Let the material of all the elements of the stiffened shell obey the Tresca yield condition and the associated flow law. Then, in the limit state of the shell, the stress in the facing layer corresponds to the points of internal hexagon shown in Fig. 1, while those in the stiffening layer correspond to the points of hexagon shown in Fig. 4.

In what follows we restrict ourselves, for simplicity, to the case, when the yield stresses for all the elements of the stiffened shell are the same under extension and compression, though different for each element.

As a reference we choose the middle facing surface. The orthogonal system of coordinates we choose such that two of its axes coincide with tangential lines towards lines of the main curvature, and the third axis $Z$ is directed along the interior normal to the surface. Then the inner tension $T_i (i = 1, 2)$. and the moments $M_i$ are connected with the main normal stresses of the shell by the following relation

$$
\begin{aligned}
2t_i &= \int_{-1}^{1} s_i \, \mathrm{d}z + \alpha\omega_i \int_{\alpha}^{\alpha(1+2h_i)} s_i' \, \mathrm{d}z, \\
m_i &= \int_{-1}^{1} s_i z \, \mathrm{d}z + \alpha\omega_i \int_{\alpha}^{\alpha(1+2h_i)} s_i' z \, \mathrm{d}z
\end{aligned}
\qquad (i = 1, 2) \qquad (2.1)
$$

Here

$$
t_i = T_i/T_0, m_i = M_i/M_0, h_i = H_i/2H, s_i = \sigma_i/\sigma_0
$$
$$
s_i' = \sigma_i'/\sigma_0, \omega_i = d_i/l_i, z = Z/H, T_0 = 2\sigma_0 H, M_0 = \sigma_0 H^2
$$

$2H$ is the facing thickness, $\sigma_0$ the yield stress of its material, $H_i$ denotes the height of ribs, in the $i$ direction, $\sigma_i$ denotes the main stresses in the facing, $\sigma_i'$ denotes the main stresses in the stiffened layer. The meaning of the quantities $d_i$, $l_i$ becomes clear from Fig. 3; $\alpha = 1$ if the stiffening layer is inside the shell; $\alpha = -1$ if it is outside. Below we admit, for definiteness, $H_2 \geq H_1$ and for convenience, that the rib density in the direction 2 is $\omega_{20}$ for $1+2h_1 \leq z \leq 1 + 2h_2$.

Then we assume that for the whole cross-section of the stiffened shell the Kirchhoff–Love hypothesis is valid. Hence, for any shell point the strain rate components $\varepsilon_i$ are expressed by the extension rates $\varepsilon_{0i}$ and curvature variation rates $K_i$ of the reference surface as follows:

$$\varepsilon_i = \varepsilon_{0i} - ZK_i \tag{2.3}$$

Then using the method developed in [5, 9] or [12] and introducing, for brevity, the notations:

$$\begin{aligned}
&t_i^{\pm} = t_i \pm t_i^*, \quad m_i^{\pm} = m_i \pm m_i^*, \quad t_1^* = \omega_1\lambda_1 h_1, \quad t_2^* = t_{20}^* + t_{21}^*,\\
&t_{20}^* = \omega_{20}\lambda_2 h_2, \quad t_{21}^* = (\omega_2 - \omega_{20})\lambda_2 h_1, \quad m_1^* = 2t_1^*(1+h_1)\\
&m_2^* = 2[t_{20}^*(1+h_2) + t_{21}^*(1+h_1)], \quad \delta_i = 1 - \frac{a_i}{\lambda_i},\\
&\delta_{12} = \omega_1 a_1 + \beta_2, \quad \delta_{21} = \omega_2 a_2 + \beta_1, \quad \beta_i = \omega_i\lambda_i,\\
&\alpha_i^{\pm} = \omega_i(\lambda_i \pm a_i)/2, \quad a_i = s_0 - \lambda_i, \quad m_2^0 = 2\alpha_2^-(1+h_1),\\
&m_{i0}^{\pm} = 4\alpha_i^{\pm} h_1(1+h_1),
\end{aligned} \tag{2.4}$$

we obtain the limit relations corresponding to the sides (Table 1) and angles (Table 2) of hexagons shown in Figs. 1 and 4.

On the left-hand side of each of these limit relations indicated are the plastic regimes effected in the shell cross-section. At that their order from left to right corresponds to the count from the free (unstiffened) cross-section of the shell edge towards its depth. The upper symbols correspond to the internal location of the stiffening layer, the lower to the external one. The parameters $l, m, n, p, q, r, u$ correspond to the boundaries, that separate the above plastic regimes over the cross-section, having the following meaning:

$$\begin{aligned}
&p = -\frac{\varepsilon_{01}}{2k_1}, \quad q = -\frac{\varepsilon_{02}}{2k_2}, \quad r = -\frac{\varepsilon_{01}+\varepsilon_{02}}{2(k_1+k_2)}, && 0 \le |z| \le 1\\
&l = -\frac{\varepsilon_{01}}{2k_1}, \quad m = -\frac{\varepsilon_{02}}{2k_2}, \quad n = -\frac{\omega_2\varepsilon_{02}+\omega_1\varepsilon_{01}}{2(\omega_2 k_2+\omega_1 k_1)} && 1 \le |z| \le 1+2h_1,\\
&u = -\frac{\varepsilon_{02}}{2k_2}, && 1+2h_1 \le |z| \le 1+2h_2
\end{aligned} \tag{2.5}$$

Here

$$k_i = -\frac{M_0 K_i}{T_0}.$$

Because parameters $r$ and $n$ in the domain $-1 \le z \le 1$ are expressed by $p, q$ and $k_1/k_2$, and in the domain $1 \le |z| \le 1+2h_1$ by $l, m$ and $k_1/k_2$, then all the parameters are expressed by three independent quantities. Therefore each of the above four equalities, listed in Table 2, is a parametric form of the finite limit relation between the quantities $t_1, t_2, m_1, m_2$. We note, that if in any of the relations of Tables 1 or 2 contain the inequality $\omega_1 \le \omega_2$, then this implies that for $\omega_1 > \omega_2$ this relation is lacking or changing. There are as many relations corresponding to regimes symmetric with respect to the origin of coordinates shown in Figs. 1 and 4, apart from those listed in Tables 1 and 2. These relations are derived from relations listed in the Tables if all the signs ($\pm$) are changed.

TABLE 1

| No. | Plastic regimes | Feature | Limit relations |
|---|---|---|---|
| $1i$ | $AB, DE, D_1E_1$ | $i = 2$ | $m_i^{\pm} \mp (t_i^{+})^2 \pm 1 = 0$ |
| | $AF, CD, C_1D_1$ | $i = 1$ | |
| $2i$ | $AB, A_1B_1, D_1E_1$ | $i = 2$ | $m_i^{\pm} \pm \beta_i \mp \frac{1}{\beta_i}(t_i^{+} + \beta_i - 1)^2 = 0$ |
| | $AF, A_1F_1, C_1D_1$ | $i = 1$ | |
| 3 | $BC, EF, E_1$ | $\omega_1 \leq \omega_2$ | $m_2^{\pm} - m_1^{\mp} \mp \delta_1 m_1^{*} \mp (t_2^{+} - t_1^{-} - \delta_1 t_1^{*})^2 \pm 1 = 0$ |
| 4 | $BC, EF, F_1$ | $\omega_1 \geq \omega_2$ | $m_2^{\pm} - m_1^{\mp} \mp \frac{2m_1^{*}\alpha_2^{-}}{\beta_1} \mp \left(t_2^{+} - t_1^{-} \mp \frac{2t_1^{*}\alpha_2^{-}}{\beta_1}\right)^2 \pm 1 = 0$ |
| 5 | $BC, B_1, E_1$ | $\omega_1 \leq \omega_2$ | $m_2^{\pm} - m_1^{\mp} \mp m_1^{*}\delta_1 \pm \delta_{12} \mp \frac{1}{\delta_{12}}(t_2^{+} - t_1^{-} - \delta_1 t_1^{*} + \delta_{12} - 1)^2 = 0$ |
| 6 | $BC, C_1, F_1$ | $\omega_1 \geq \omega_2$ | $m_2^{\pm} - m_1^{\mp} \mp \frac{2m_1^{*}\alpha_2^{-}}{\beta_1} \pm \delta_{21} \mp \frac{1}{\delta_{21}}\left(t_2^{+} - t_1^{-} - \frac{2t_1^{*}\alpha_2^{-}}{\beta_1} + \delta_{21} - 1\right)^2 = 0$ |
| 7 | $B, E, E_1F_1$ | $\omega_1 \leq \omega_2$ | $\omega_1 m_2^{\pm} - \omega_2 m_1^{\mp} \mp \delta_1 \omega_2 m_1^{*} \mp \frac{1}{\omega_1}(\omega_1 t_2^{+} - \omega_2 t_1^{-} - \delta_1 \omega_2 t_1^{*})^2 \mp \omega_1 = 0$ |
| 8 | $F, C, C_1B_1$ | $\omega_1 \geq \omega_2$ | $\omega_1 m_2^{\mp} - \omega_2 m_1^{\pm} \pm \delta_1 \omega_2 m_1^{*} \pm \frac{1}{\omega_2}(\omega_1 t_2^{-} - \omega_2 t_1^{+} + \delta_1 \omega_2 t_1^{*})^2 \pm \omega_2 = 0$ |
| 9 | $C, C_1B_1, E_1F_1$ | $\omega_1 \leq \omega_2$ | $\omega_1 m_2^{\pm} - \omega_2 m_1^{\mp} \mp \delta_1 \omega_2 m_1^{*} \pm \omega_1 \omega_2 \tau$ $\mp \frac{1}{\omega_1 \omega_2 \tau}(\omega_1 t_2^{+} - \omega_2 t_1^{-} - \delta_1 \omega_2 t_1^{*} + \omega_1 \omega_2 \tau - \omega_2)^2 = 0$ |
| 10 | $B, C_1B_1, E_1F_1$ | $\omega_1 \geq \omega_2$ | $\omega_1 m_2^{\pm} - \omega_2 m_1^{\mp} \mp \delta_1 \omega_2 m_1^{*} \pm \omega_1 \omega_2 \tau$ $\mp \frac{1}{\omega_1 \omega_2 \tau}(\omega_1 t_2^{+} - \omega_2 t_1^{-} - \delta_1 \omega_2 t_1^{*} + \omega_1 \omega_2 \tau - \omega_1)^2 = 0$ |

It can be easily checked, that if the left-hand side of each of these relations is denoted by $f(t_1, t_2, m_1, m_2)$, then the flow law for them is defined by the relation:

$$\varepsilon_{01}:\varepsilon_{02}:k_1:k_2 = \frac{\partial f}{\partial t_1}:\frac{\partial f}{\partial t_2}:\frac{\partial f}{\partial m_1}:\frac{\partial f}{\partial m_2}. \tag{2.6}$$

As an example, we define the limit relation corresponding to regimes $AB, DE, D_1E_1$ when the stiffening layer has internal location. Let in the domain $-1 \leq z \leq p$ of the facing cross-section the state $AB$ (Fig. 1) be realized, for which

$$s_2 = 1, 0 \leq s_1 \leq 1, \varepsilon_2 > 0, \varepsilon_1 = 0 \, (-1 \leq z \leq p) \tag{2.7}$$

and in accordance with the flow law and equality (2.3) $\varepsilon_{01} - zk_1 = 0$. Since in the above interval the quantity $z$ is arbitrary, then this equality is equivalent to $\varepsilon_{01} = 0$, $k_1 = 0$. Hence for all $z \geq -1$ the strain rate is $\varepsilon_1 = 0$. Then, in a general case, taking into account

TABLE 2

| No. | Plastic regimes | Feature | Limit relations |
|---|---|---|---|
| 1 | $^{\mp 1}A^p, B^r, C^q, D^{\pm 1}$ $D_1^{\pm(1+2h_1)}, -\lambda_2^{\pm(1+2h_2)}$ | | $t_1^+ = \pm\frac{p+r}{2}$, $t_2^+ = \pm\frac{r+q}{2}$ <br> $m_1^\pm = \pm\frac{p^2+r^2}{2}\mp 1$, $m_2^\pm = \pm\frac{r^2+q^2}{2}\mp 1$ |
| 2 | $^{\mp 1}A^{\pm 1}, A_1^l, B_1^n, C_1^m, D_1^{\pm(1+2h_1)}$, $-\lambda_2$ | | $t_1^+ = 1-\beta_1\pm\alpha_1^+ l\pm\alpha_1^- n$, <br> $t_2^+ = 1-\beta_2\pm\alpha_2^+ m\pm\alpha_2^- n$, <br> $m_1^\pm = \mp\beta_1\pm\alpha_1^+ l^2\pm\alpha_1^- n^2$, <br> $m_2^\pm = \mp\beta_2\pm\alpha_2^+ m^2\pm\alpha_2^- n^2$ |
| 3 | $A^p, B^r, C^{\pm 1}, B_1^n, C_1^m, D_1^{\pm(1+2h_1)}$, $-\lambda_2$ | $\omega_1\le\omega_2$ | $t_1^+ = \pm\frac{p+r}{2}\pm\alpha_1^-(n\pm 1)$, <br> $t_2^+ = \frac{1\pm r}{2}-\beta_2\pm\alpha_2^- n\pm\alpha_2^+ m$, <br> $m_1^\pm = \mp 1\pm\frac{p^2+r^2}{2}\pm\alpha_1^-(n^2-1)$, <br> $m_2^\pm = \pm\frac{r^2-1}{2}\mp\beta_2\pm\alpha_2^- n^2\pm\alpha_2^+ m^2$, |
| 4 | $A^q, F^r, E^{\pm 1}, F_1^n, E_1^l, D_1^{\pm(1+2h_1)}$, $-\lambda_2$ | $\omega_1\ge\omega_2$ | $t_1^+ = \frac{1\pm r}{2}-\beta_1\pm\alpha_1^- n\pm\alpha_1^+ l$, <br> $t_2^+ = \pm\frac{q+r}{2}\pm\alpha_2^-(n\mp 1)$, <br> $m_1^\pm = \pm\frac{r^2-1}{2}\mp\beta_1\pm\alpha_1^- n^2\pm\alpha_1^+ l^2$, <br> $m_2^\pm = \pm\frac{q^2+r^2}{2}\pm\alpha_2^-(n^2-1)\mp 1$ |
| 5 | $B^r, C^q, D^{\pm 1}, D_1^l, E_1^n, F_1, -\lambda_2$ | | $t_1^- = -\frac{1\mp r}{2}+\beta_1\mp\alpha_1^+ l\mp\alpha_1^- n$, <br> $t_2^+ = \pm\frac{r+q}{2}\mp\alpha_2^-(n\mp 1)+\alpha_2^- h_1$, <br> $m_1^\mp = \mp\frac{1-r^2}{2}\pm\beta_1\mp\alpha_1^+ l^2\mp\alpha_1^- n^2$, <br> $m_2^\pm = \pm\frac{r^2+q^2}{2}\mp\alpha_2^-(n^2-1)\pm m_2^0\mp 1$ |
| 6 | $B^r, C^q, D^p, E^{\pm 1}, E_1^{\pm(1+2h_1)}, -\lambda_2$ | | $t_1^- = \frac{r-p}{2}-2h_1\alpha_1^-$, $t_2^+ = \pm\frac{r+q}{2}$, <br> $m_1^\mp = \mp\frac{p^2-r^2}{2}\mp m_{10}^-$, <br> $m_2^\pm = \pm\frac{r^2+q^2}{2}\mp m_{20}^-\pm m_2^0\mp 1$ |

TABLE 2. Continued

| No. | Plastic regimes | Feature | Limit relations |
|---|---|---|---|
| 7 | $B^r, C^{\pm 1}, B_1^n, C_1^m, D_1^l, E_1, -\lambda_2$ | $\omega_1 \leq \omega_2$ and $n = \pm 1$ at $\omega_1 \geq \omega_2$ | $t_1^- = -\frac{1\mp r}{2} + \alpha_1^+(1\mp l) - \alpha_1^-(1+2h_1\mp n)$,<br>$t_2^+ = \frac{1\pm r}{2} - \alpha_2^+(1\mp m) - \alpha_2^-(1\mp n)$,<br>$m_1^\mp = \mp\frac{1-r^2}{2} \pm \alpha_1^+(1-l^2) \pm \alpha_1^-(n^2-1) \mp m_{10}^-$,<br>$m_2^\pm = \mp\frac{1-r^2}{2} \mp \alpha_2^+(1-m^2) \pm \alpha_2^-(n^2-1) \mp m_{20}^- \pm m_2^0$ |
| 8 | $B^p, A^q, F^r, E^{\pm 1}, F_1^n, E_1, -\lambda_2$ | $\omega_1 \geq \omega_2$ and $n = \pm 1$ at $\omega_1 \leq \omega_2$ | $t_1^- = \frac{r-p}{2} + \alpha_1^-(n\mp 1) - \delta_1 t_1^*$,<br>$t_2^+ = \frac{q+r}{2} + \alpha_2^-(n\mp 1)$,<br>$m_1^\mp = \pm\frac{r^2-p^2}{2} \pm \alpha_1^-(n^2-1) \mp \delta_1 m_1^*$,<br>$m_2^\pm = \pm\frac{q^2+r^2}{2} \pm \alpha_2^-(n^2-1) \mp 1$ |
| 9 | $B^p, A^{\pm 1}, A_1^m, F_1^n, E_1, -\lambda_2$ | | $t_1^- = \frac{1\mp p}{2} \pm \alpha_1^-(n\mp 1) - \delta_1 t_1^*$,<br>$t_2^+ = 1 - \beta_2 \pm \alpha_2^- n \pm \alpha_2^+ m$,<br>$m_1^\mp = \mp\frac{p^2+1}{2} \pm \alpha_1^-(n^2-1) \mp \delta_1 m_1^*$,<br>$m_2^\pm = \mp\beta_2 \pm \alpha_2^- n^2 \pm \alpha_2^+ m^2$ |
| 10 | $B^{\pm 1}, B_1^l, A_1^m, F_1^n, E_1, -\lambda_2$ | | $t_1^- = \omega_1 a_1 \pm \alpha_1^- n \mp \alpha_1^+ l - \delta_1 t_1^*$,<br>$t_2^+ = 1 - \beta_2 \pm \alpha_2^+ m \pm \alpha_2^- n$,<br>$m_1^\mp = \pm\omega_1 a_1 \pm \alpha_1^- n^2 \mp \alpha_1^+ l^2 \mp \delta_1 m_1^*$,<br>$m_2^\pm = \mp\beta_2 \pm \alpha_2^- n^2 \pm \alpha_2^+ m^2$ |
| 11 | $C^r, B^p, A^q, F^{\pm 1}, F_1, -\lambda_2$ | | $t_1^- = \mp\frac{r+p}{2} - \delta_1 t_1^* + 2h_1\alpha_1^-$, $t_2^+ = \pm\frac{q-r}{2} + 2h_1\alpha_2^-$,<br>$m_1^\mp = \mp\frac{r^2+p^2}{2} \mp \delta_1 m_1^* \pm m_{10}^-$, $m_2^\pm = \pm\frac{q^2-r^2}{2} \pm m_{20}^-$ |
| 12 | $C^r, B^p, A^{\pm 1}, A_1^m, F_1, -\lambda_2$ | | $t_1^- = \mp\frac{r+p}{2} - \delta_1 t_1^* + 2h_1\alpha_1^-$,<br>$t_2^+ = \frac{1\mp r}{2} - \beta_2 \pm \alpha_2^+ m + 2\alpha_2^- h_1 + \alpha_2^-$,<br>$m_1^\mp = \mp\frac{r^2+p^2}{2} + 1 \mp \delta_1 m_1^* \pm m_{10}^-$,<br>$m_2^\pm = \pm\frac{1-r^2}{2} \pm \alpha_2^+ m^2 \mp \beta_2 \pm \alpha_2^- \pm m_{20}^-$ |

TABLE 2. Continued

| No. | Plastic regimes | Feature | Limit relations |
|---|---|---|---|
| 13 | $C^r, B^{\pm 1}, C_1^n, B_1^l, A_1^m, F_1, -\lambda_2$ | $\omega_1 \geq \omega_2$ and $n = \pm 1$ at $\omega_1 \leq \omega_2$ | $t_1^- = -\frac{1 \pm r}{2} + \alpha_1^+(1 \mp l) \mp \alpha_1^-(n \mp 1 \mp 2h_1) - \delta_1 t_1^*$, <br> $t_2^+ = \frac{1 \mp r}{2} - \alpha_2^+(1 \mp m) \mp \alpha_2^-(n \mp 1 \mp 2h_1)$, <br> $m_1^\mp = \mp \frac{r^2 - 1}{2} \pm \alpha_1^+(1 - l^2) \mp \alpha_1^-(n^2 - 1) \pm m_{10}^- \mp \delta_1 m_1^*$, <br> $m_2^\pm = \pm \frac{1 - r^2}{2} \mp \alpha_2^+(1 - m^2) \mp \alpha_2^-(n^2 - 1) \pm m_{20}^-$ |
| 14 | $C^q, D^p, E^r, F^{\pm 1}, E_1^n, F_1, -\lambda_2$ | $\omega_1 \leq \omega_2$ and $n = \pm 1$ at $\omega_1 \geq \omega_2$ | $t_1^- = \mp \frac{p + r}{2} \mp \alpha_1^-(n \mp 1), t_2^+ = \mp \frac{r - q}{2} \mp \alpha_2^-(n \mp 1)$ <br> $+ 2\alpha_2^- h_1$, <br> $m_1^\mp = \mp \frac{p^2 + r^2}{2} \mp \alpha_1^-(n^2 - 1) \pm 1$, <br> $m_2^\pm = \mp \frac{r^2 - q^2}{2} \mp \alpha_2^-(n^2 - 1) \pm m_2^0$ |
| 15 | $C^q, D^{\pm 1}, D_1^l, E_1^n, F_1, -\lambda_2$ | | $t_1^- = -1 + \beta_1 \mp \alpha_1^- n \mp \alpha_1^+ l$, <br> $t_2^+ = -\frac{1 \mp q}{2} \mp \alpha_2^-(n \mp 1) + 2\alpha_2^- h_1$, <br> $m_1^\mp = -\beta_1 \mp \alpha_2^+ l^2 \mp \alpha_1^- n^2$, <br> $m_2^\pm = \mp \frac{1 - q^2}{2} \mp \alpha_2^-(n^2 - 1) \pm m_2^0$ |
| 16 | $C^{\pm 1}, C_1^m, D_1^l, E_1^n, F_1, -\lambda_2$ | | $t_1^- = -1 + \beta_1 \mp \alpha_1^+ l \mp \alpha_1^- n$, <br> $t_2^+ = -\omega_2 a_2 \pm \alpha_2^+ m \mp \alpha_2^- n + 2\alpha_2^- h_1$, <br> $m_1^\mp = \pm \beta_1 \mp \alpha_1^+ l^2 \mp \alpha_1^- n^2$, <br> $m_2^\pm = \mp \omega_2 a_2 \mp \alpha_2^- n^2 \pm \alpha_2^+ m^2 \mp m_2^0$ |

the flow law for the yield conditions corresponding to hexagons shown in Figs. 1 and 4 we shall have the following distribution of stresses in the remaining part of the shell cross-section:

$$\begin{aligned} s_2 &= -1, \quad & -1 \leq s_1 \leq 0, \quad & q \leq z \leq 1, \\ s_2' &= \lambda_2, \quad & -\lambda_1 \leq s_1' \leq a_1, \quad & 1 \leq z \leq 1 + 2h_1, \\ s_2' &= \lambda_2, \quad & s_1' = 0, \quad & 1 + 2h_1 \leq z \leq 1 + 2h_2 \end{aligned} \tag{2.8}$$

Since $\varepsilon_{01} = 0$, $k_1 = 0$, then quantities $t_1, m_1$ play the role of reactions and would not be present in the limit relation. Therefore, by substituting the stresses (2.7) and (2.8) in expressions (2.1) by $t_2$ and $m_2$ at $\alpha = 1$ and after integrating having considered symbols in (2.4), we have:

$$t_2^+ = q, \qquad m_2^+ + 1 = q^2 \tag{2.9}$$

or

$$f(t_1, t_2, m_1, m_2) = m_2^+ - (t_2^+)^2 + 1 = 0 \tag{2.10}$$

Since in this case $\varepsilon_{01} = k_1 = 0$, $q = -\varepsilon_{02}/2k_2$ then (2.9) and (2.10) imply, that the flow law is fulfilled as (2.6). For exterior location of the ribs ($\alpha = -1$) we obtain the limit relation for the same limit regimes $AB$, $DE$, $D_1E_1$ in the shell cross-section

$$f(t_1, t_2, m_1, m_2) = m_2^- + (t_2^+)^2 - 1 = 0 \tag{2.11}$$

On changing the signs in (2.11) and (2.10) we derive the limit relations:

$$f(t_1, t_2, m_1, m_2) = m_2^\mp \pm (t_2^-)^2 \mp 1 = 0$$

which are not contained in Table 1 and correspond to the symmetric regimes $DE$, $AB$, $A_1B_1$. Similarly, one can obtain all the remaining relations omitted in Table 1 to make sure, that the flow law is fulfilled in all cases (2.6). To explain the form and the method of derivation of corresponding limit relations accepted in Table 2, we consider, for example, the case, when at the interior location of the stiffening layer the plastic regimes of the facing are effected in the shell cross-section $A, B, C, D, D_1, -\lambda_2$.

Then the stress distribution in the shell cross-section has the form:

$$\begin{array}{lll}
A & s_1 = s_2 = 1, & -1 \le z \le p \\
B & s_1 = 0,\ s_2 = 1, & p \le z \le r \\
C & s_1 = -1,\ s_2 = 0, & r \le z \le q \\
D & s_1 = s_2 = -1, & q \le z \le 1 \\
D_1 & s_1' = -\lambda_1,\ s_2' = -\lambda_2, & 1 \le z \le 1+2h_1 \\
 & s_1' = 0,\ s_2' = -\lambda_2, & 1+2h_1 \le z \le 1+2h_2
\end{array}$$

Substituting these values in (2.1) (at $\alpha = 1$) we obtain on integrating:

$$\begin{gathered} t_1^+ = \tfrac{1}{2}(p+r),\ t_2^+ = \tfrac{1}{2}(r+q),\ m_1^+ = -1 + \tfrac{1}{2}(p^2+r^2), \\ m_2^+ = -1 + \tfrac{1}{2}(r^2+q^2) \end{gathered} \tag{2.12}$$

In each particular case one can exclude the parameters and represent corresponding limit relations in explicit form. Then from constraints on the parameters we obtain corresponding restrictions for the limit relations expressed by $t_1, t_2, m_1, m_2$. For example, in the case under consideration, from (2.12) we have:

$$\begin{gathered} q = \frac{m_2^+ - m_1^+}{2(t_2^+ - t_1^+)} + t_2^+ - t_1^+, \qquad p = \frac{m_2^+ - m_1^+}{2(t_2^+ - t_1^+)} - t_2^+ + t_1^+ \\ r = -\frac{m_2^+ - m_1^+}{2(t_2^+ - t_1^+)} + t_2^+ + t_1^+. \end{gathered} \tag{2.13}$$

By substituting these values in the latter equations from (2.12) after simple transformations we have:

$$m_2^+ + 1 - (t_2^+)^2 - (t_1^+)^2 - \frac{m_2^+ - m_1^+}{2(t_2^+ - t_1^+)}\left[\frac{m_2^+ - m_1^+}{2(t_2^+ - t_1^+)} - 2t_1^+\right] = 0 \tag{2.14}$$

But as in the case under consideration $-1 \le p \le r \le q \le 1$ then this limit relation is effected when the following inequalities are observed

$$m_2^+ - m_1^+ - 2t_2^+(t_2^+ - t_1^+) \le 0, \qquad m_2^+ - m_1^+ - 2t_1^+(t_2^+ - t_1^+) \ge 0,$$
$$m_2^+ - m_1^+ + 2(t_2^+ - t_1^+)(1 + t_1^+ - t_2^+) \ge 0,$$
$$m_2^+ - m_1^+ - 2(t_2^+ - t_1^+)(1 + t_1^+ - t_2^+) \le 0, \qquad t_2^+ - t_1^+ \ge 0.$$

The flow law for the limit relations obtained when one realized the angular regime in the shell cross-section is defined in accordance with the meaning of parameters (2.5). In each particular case one can check, that, if on excluding parameters, the limit state takes the form:

$$f(t_1, t_2, m_1, m_2) = 0$$

then the flow law is defined by equalities (2.6). Indeed, if, for example, in the case under consideration for the totality of plastic regimes $A, B, C, D, D_1, -\lambda_2$ the left-hand side of equation (2.14) is denoted by $f(t_1, t_2, m_1, m_2)$, then using equalities (2.5) and (2.13), one can easily see the validity of relation (2.6).

We note the following characteristic features of the limit relations obtained. As mentioned above, in the stress space the yield condition is expressed as a piecewise linear hexagon and the associated flow law corresponds to the strain rate vector orthogonality condition with respect to the sides of these hexagons but not to all of them. The limit relations obtained in the space of generalized tensions $t_1, t_2, m_1, m_2$ determine the closed convex hypersurface which is symmetric with respect to the origin of coordinates constructing the nonlinear "pieces". The flow law expressed by the generalized strain rates $\varepsilon_{01}, \varepsilon_{02}, k_1, k_2$ corresponds to the orthogonality condition of the strain rate vector for all "pieces" of this surface. Finally, the limit relations in general, depend on the coordinate of the meridian as such a dependence can be possessed by quantities $\omega_i$ entering them. Because of the specific nature of the flow law for $B_1C_1$ and $E_1F_1$ sides of hexagon shown in Fig. 4, at $\omega_1 = \omega_2$ all the limit relations of the second table are not altered. At $\omega_1 < \omega_2$ in relations

$$A, F, E, F_1, E_1, D_1, -\lambda_2; \qquad B, A, F, E, F_1, E_1, -\lambda_2;$$
$$C, B, C_1, B_1, A_1, F_1, -\lambda_2$$

of table 2 we put $n = \pm 1$ (superscript denotes shells, stiffened from inside, the subscript denotes shells stiffened from outside) and for $\omega_1 > \omega_2$ in relation

$$B, C, B_1, C_1, D_1, E_1, -\lambda_2; C, D, E, F, E_1, F_1, -\lambda_2;$$
$$A, B, C, B_1, C_1, D_1, -\lambda_2.$$

we put $n = \pm 1$.

## 3. PARTICULAR CASES OF LIMIT RELATIONS

The limit relations obtained represent a full system of limit relations for the stiffened waffle-type shells. The hexagon $A_1B_1C_1D_1E_1F_1$ shown in Fig. 4 degenerates into tetragon $A_1H_1D_1G_1$ at $a_1 \to -\lambda_1$ and $a_2 \to \lambda_2$. Then by putting these values for $a_1$ and $a_2$, and not applying in Table 1 by relations containing regimes $B_1C_1$, $E_1F_1$ and by equating in Table 2 the parameters which separate regimes $B_1$, $C_1$ and $E_1$, $F_1$ we obtain the limit relations for the ribbed shells.

By putting in relations of the preceding paragraph $\omega_1 = \omega_2 = 1$, $\omega_{20} = 0$, $h_1 = h_2$ we obtain the limit relation for the bimetal shells. For $h_1 = h_2 = 0$ the latter degenerates into the known [12] limit relation for the homogeneous isotopic shells.

For the case $\omega_1 = 0$, $\omega_2 \neq 0$ or $\omega_2 = 0$, $\omega_1 \neq 0$ we obtain the limit relations for shells stiffened by ribs of one direction.

Below we admit that directions 1 and 2 coincide, respectively, with meridional and circumferential directions. Then for cylindrical shells we have $k_2 = 0$ and in accordance with equality (2.5) $|q| \to \infty$, $|m| \to \infty$, $|u| \to \infty$. As at $|z| \geq 1$ the facing material is lacking at $|z| \geq 1+2h_1$ the stiffening layer is lacking, and at $|z| \geq 1+2h_2$ the rib material is lacking in the circumferential direction; then to obtain the limit relations for cylindric shells it is necessary to put $|q| = 1$, $|m| = 1+2h_1$, $|u| = 1+2h_2$, in all the relations of section 2 aside from equality $k_2 = 0$. Moreover, if in the domain $-1 \leq z \leq 1$ either of parameters $p$ or $r$ exceeds $q$ of the module, then its value must be put equal to value $q$. Similarly, if in the domain $1 \leq |z| \leq 1+2h_1$ either of the parameters $l$ or $n$ exceeds $m$ of the module, then its value should be put equal to $m$ value. Then taking into account that in the case under consideration $m_2$ plays the role of reaction, so it would be contained in the limit relation, after doing simple transformation we get the following limit relations for cylindric waffle-type shells [9]

$(AB, A_1B_1)$:

$$t_2^- = 1, \tag{3.1}$$

$(BC, C_1)$:

$$t_2^- - t_1^+ = 1 - v_2^- h_1, \qquad (\omega_1 \geq \omega_2) \tag{3.2}$$

$(B, B_1C_1)$:

$$\omega_1 t_2^- - \omega_2 t_1 = \omega_1, \qquad (\omega_1 \geq \omega_2) \tag{3.3}$$

$(BC, B_1)$:

$$t_2^- - t_1^+ = 1 - v_1^+ h_1, \qquad (\omega_1 \leq \omega_2) \tag{3.4}$$

$(C, B_1C_1)$:

$$\omega_1 t_2^- - \omega_2 t_1^+ = \omega_2[1 - \omega_1 h_1(\lambda_2 - \mu_2)], \qquad (\omega_1 \leq \omega_2) \tag{3.5}$$

$(A, B, B_1, C_1)$:

$$\frac{2m_1^+}{1+v_1^+} = -1 + \left(\frac{2t_1^+ + v_1^+ - 1}{1+v_1^+}\right)^2 + 4v_1^+ \left[\frac{t_2^- + v_2^- h_1 - 1}{v_2^-} - \frac{t_1^+ - 1}{1+v_1^+}\right]^2, \tag{3.6}$$

$(A, A_1, B_1, C_1)$:

$$\frac{m_1^+}{\beta_1} = -1 + \left(\frac{t_1^- - 1}{\beta_1} + 1 + 2h_1\right)^2 + \frac{v_1^+}{v_1^-}\left[\frac{2(t_2^- - 1)}{v_2^-} - \frac{t_1^- - 1}{\beta_1}\right]^2, \tag{3.7}$$

$(A, B, C, C_1)$:

$$m_1^+ = -1 + (t_1^+)^2 + 4(t_2^- + v_2^- h_1 - \tfrac{1}{2} + \tfrac{1}{2}t_1^+)^2, \tag{3.8}$$

$(C, C_1, B_1, A_1)$:

$$\frac{m_1^-}{\beta_1} = 1-\left(\frac{t_1^++1}{\beta_1}-1-2h_1\right)^2-\frac{v_1^+}{v_1^-}\left[\frac{2t_2^-}{v_2^-}+2h_1-\frac{t_1^++1}{\beta_1}\right]^2, \tag{3.9}$$

$(C, B, B_1, A_1)$:

$$\frac{2m_1^-}{1+v_1^-} = 1-\left(\frac{2t_1^- - v_1^- +1}{1+v_1^-}\right)^2-\frac{4}{v_1^-}\left(\frac{t_1^- - v_1^-}{1+v_1^-}+1-t_2^-\right)^2, \tag{3.10}$$

$(C, B, A, A_1)$:

$$m_1^- = 1-(t_1^-)^2-(2t_2^- - t_1^- - 1)^2, \tag{3.11}$$

$(AF, CD, C_1D_1)$:

$$m_1^+ = -1+(t_1^+)^2, \tag{3.12}$$

$(AF, A_1F_1, C_1D_1)$:

$$\frac{m_1^+}{\beta_1} = -1+\left(\frac{t_1^- - 1}{\beta_1}+1+2h_1\right)^2, \tag{3.13}$$

$(CD, AF, A_1F_1)$:

$$m_1^- = 1-(t_1^-)^2, \tag{3.14}$$

$(CD, C_1D_1, A_1F_1)$:

$$\frac{m_1^-}{\beta_1} = 1-\left(\frac{t_1^++1}{\beta_1}-1-2h_1\right)^2. \tag{3.15}$$

Here aside from the already used notations we also used the following notations: $v_i^\pm = \omega_i(\lambda_i \pm \mu_i)$, $\mu_1 = 0$, $\mu_2 = \lambda_2 - \lambda_1$, $\alpha = 1$ denotes internal location of stiffening layer, $\alpha = -1$ denotes the external one. The plastic regimes on the left in relations (3.1)–(3.15) are computed from the edge $z = -1$ of the shell cross-section in the direction of increasing $z$.

## 4. APPROXIMATE LIMIT RELATIONS

In the limit state of the shell, the tensions in the shell element must satisfy not only relations of section 2, but also the equation of equilibrium [1]

$$\begin{aligned} (r_0t_1)' - r_1t_2\cos\varphi - r_0q_1 + r_0r_1p_1 &= 0, \\ r_1t_2\sin\varphi + r_0t_1 + (r_0q_1)' + r_0r_1p_3 &= 0, \\ (r_0m_1)' - r_1m_2\cos\varphi - r_0r_1q_1T_0L/M_0 &= 0. \end{aligned} \tag{4.1}$$

Here $r_i = R_i/L$ denotes dimensionless radii of curvature; $r_0 = R_0/L$ the dimensionless radius of the parallel line; $p_1 = P_1L/T_0$, $p_3 = P_3L/T_0$ the dimensionless components of load intensity in the meridian and normal directions, respectively; $q_1 = Q_1/T_0$ denotes the dimensionless shearing force, $L$ denotes the characteristic shell length. A prime indicates differentiation over $\varphi$ the angle between the normal and the rotation axis. The generalized strains, aside from relations (2.6), must also satisfy the following kinematic relations

$$\varepsilon_{01} = \frac{v'-w}{r_1}, \qquad \varepsilon_{02} = \frac{v\,\mathrm{ctg}\,\varphi - w}{r_2}, \qquad k_1 = \frac{M_0}{T_0 L r_1}\left(\frac{v+w'}{r_1}\right)',$$
$$k_2 = \frac{M_0}{T_0 L}\frac{v+w'}{r_1}\frac{\mathrm{ctg}\,\varphi}{r_2}, \tag{4.2}$$

$v$, $w$ denotes dimensionless displacements in the meridian and the normal direction.

Solution of the problem of plastic analysis of stiffened axisymmetric shells is connected with integration of the set of equations (4.1)–(4.2), together with the limit relations of section 2 and the associated flow law (2.6), with appropriate boundary condition.

A great number of limit relations makes problem solution practically unrealizable, in general, not only due to the nonlinearity of these relations but also (mainly) because it is not known *a priori* in which part of the shell this relation is realized. In this connection there arises a problem of using an approximate approach to the problem, involving as few limit relations as possible.

One of such approaches is to make use of only the limit relations of Table 1. As a result, we shall have a convex hypersurface in the 4-dimensional space of generalized tensions which coincides partially with the exact hypersurface and partially described with respect to it. This is likely to give most accurate and most convenient approach to the problem solution as the limit relation of Table 1 are practically all the same, which permits to construct integrals of the problem for the general case. We shall omit all the details of such an approach and note that since the approximate hypersurface in this case is described with respect to the accurate one, the limit load resulting from this approach will determine the upper boundary of the real limit load.

Due to the loss of exactness of solution further simplification of the problem is possible if we gradually refute from some relations in Table 1. The problem of choosing some of relations from Table 1 must be decided separately, depending on the specific nature of a problem and geometric parameters of the stiffening layer.

Another possible approach to the problem is linearization of the limit relation at the expense of use made of the layer models of stiffened shells. Some of the results in this direction were obtained in [5–7]. That analysis shows that when applying layer models there it is necessary to restrict ourselves to a reasonable limit since the increase of model layer number can eliminate simplification due to linearization. Thus in [7] it was shown that if $n$ denotes the layer number of ribbed shell models, then the limit polyhedron contains $12(n-1)$ faces. So it seems to be possible, within reasonable limits, to use a two-layer model only where one layer models the facing while the other models the stiffening layer.

In this case it is convenient to choose as a reference surface that of the facing layer and stiffening layer. Then by choosing the model parameters such that the rigidity of its elements would coincide in extension and bending with those of the modeled elements, and using the same methods as in refs. [1, 5–7], we obtain the following set of linearialized limiting relations

$$h_1(t_2^+ - t_1^+) \mp (m_2^\pm - m_1^\pm) = 1 + h_1,$$
$$h_1 t_i^+ \mp m_i^\pm = 1 + h_1, \qquad t_i^- \pm m_i^\mp = 0, \qquad (i = 1, 2)$$
$$(h_2 + h_1)t_2^- \mp m_2^\mp = 1 + h_1 + h_2, \tag{4.3}$$
$$(\omega_1 t_2^- - \omega_2 t_1^-) \pm (\omega_1 m_2^\mp - \omega_2 m_1^\mp) = \omega_1 c_2(1 + h_1),$$
$$c_2 = \omega_2 h_1(s_0 - \lambda_2 + \lambda_1)$$

and the same number of relations by changing all the signs. Here the notations (2.1) and (2.4) are used, but $m_i = M_i/2\sigma_0 H^2$.

The superscripts correspond to the internal location of the stiffening layer, the subscripts to its external location. In the space of the generalized stresses $t_i, m_i$ $(i = 1, 2)$ the limiting relations determine a convex 14-hedron. If we denote the left side of each of the equations (4.3) as $f(t_1, t_2, m_1, m_2)$ then the flow law for its faces is determined by the equality (2.6). The flow law for the edges of this polyhedron can be determined just as it is done for the angles of the Tresca hexagon [1, 14].

If $\omega_1 = \omega_2 = 1, \lambda_1 = \lambda_2$, then from (4.3) we will obtain the linearized limiting relations for bimetallic shells strengthened by one-way ribs. Besides, if $\lambda_1 = \lambda_2 = 1$, $\omega_{20} = 0$, $h_1 = h_2 = 1$, then discarding all but the last relation in (4.3) we will obtain the limiting relations for the uniform isotropic shells [1].

Discarding in (4.3) the last equation, we will have the linearized limiting relations for ribbed axisymmetric shells. For cylindrical ribbed shells similar relations have been obtained in [5, 7].

Instead of (4.3) one can use another system of approximate linearized relations:

$$t_i^- = 1, \qquad m_i^- = 1, \qquad (i, j = 1, 2, \quad i \neq j)$$

$$\left.\begin{aligned} t_2^- - t_1^+ &= 1 + d_i, \qquad m_2^+ - m_1^- = -1 + d_i h_1 \\ \omega_1 t_2^- - \omega_2 t_1^+ &= \omega_i(1 + c_j), \qquad \omega_1 m_2^- - \omega_2 m_1^- = \omega_i(1 + c_j h_1) \end{aligned}\right\} \omega_i \leq \omega_j \tag{4.4}$$

$$d_i = \omega_i h_1(a_i - \lambda_i), \qquad c_i = \omega_i h_1(s_0 - \lambda_2 + \lambda_1).$$

And the same number of relations must be added to the above ones by changing all signs in them.

If $\lambda_1 = \lambda_2 = 1, \omega_{20} = 0, h_1 = 1$, relations (4.4) coincide with the approximate relations for the smooth isotropic shells, obtained in ref. [15].

If $\omega_1 = \omega_2 = 1, \omega_{20} = 0, \lambda_1 = \lambda_2$ we will obtain the limiting relations of the bimetallic shells and if $\omega_{20} = \omega_2$, then we discard the equations of the last line in (4.4) and obtain the limiting relations for ribbed shells.

Note that relations (4.4) are characterized by their independence from the position of the strengthening layer.

## 5. LOAD-CARRYING CAPACITY OF STRENGTHENED CYLINDRICAL SHELLS

As an example of applying the obtained relations, consider the problem of the load-carrying capacity of a strengthened cylindrical shell loaded with an internal uniformly distributed pressure $P_1$. For definiteness we assume the shell to be free at the edge $x = 0$ and clamped at the edge $x = 2L$, $2L$ being the length of the shell. In the adopted notation the shell equilibrium equations have the form

$$m_1'' + 2\mu^2(t_2 - p_1) = 0, \; t_1 = 0. \tag{5.1}$$

Here $\mu^2 = L^2/RH$, $p_1 = P_1 R/T_0$, $R$ is the shell radius, the prime designates the derivative on a non-dimensional coordinate $y = x/L$. It is obvious that for this problem $t_2 > 0$. Hence, further it is insufficient to use a limiting curve in the first quadrant of the plane $m_1, t_2$.

For waffle-type shells in case the node is formed by the ring ribs this limiting curve takes the form represented in Fig. 5. Here the parabolas $bc$ and $cd$ at $t_1^* \leq 1$ ($t_1 = 0$) are determined, respectively, from equations (3.11) and (3.10) in case of internal position of the strengthening layer ($\alpha = 1$) and from equations (3.8) and (3.6) in case of its external position ($\alpha = -1$).

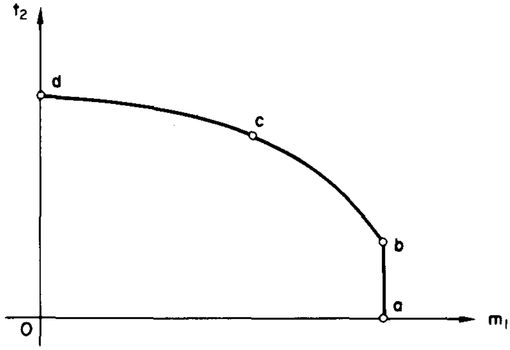


FIG. 5.

The abscissa $m_1^b$ of the point $b$ in Fig. 5 is determined from equation (3.14) at $t_1 = 0$, $\alpha = 1$ or from equation (3.12) at $t_1 = 0, \alpha = -1$.

If $t_1^* \geq 1$, the parabolas $bc$ and $cd$ are determined, respectively, from equations (3.9) and (3.10) in case of internal position of the strengthening layer ($\alpha = 1$, $t_1 = 0$) and from equations (3.7) and (3.6) in case of its external position ($\alpha = -1$, $t_1 = 0$). The abscissa $m_1^b$ is determined from equation (3.15) at $\alpha = 1$, $t_1 = 0$ or from equation (3.13) at $\alpha = -1$, $t_1 = 0$.

Let us present the equations of the parabolas $bc$ and $cd$ in the form

$$
\begin{aligned}
bc\colon t_2^- &= b_1 + (b_2 + b_3 m_1)^{\frac{1}{2}}, \\
cd\colon t_2^- &= b_4 + (b_5 + b_6 m_1)^{\frac{1}{2}}.
\end{aligned} \tag{5.2}
$$

Then, substituting the values of (5.2) into (5.1) and integrating the obtained equation at the boundary conditions

$$m_1|_{y=0} = m_1'|_{y=0} = 0 \tag{5.3}$$

and the continuity conditions

$$m_1|_{y=y_1-0} = m_1|_{y=y_1+0} = m_1^c, m_1'|_{y=y_1-0} = m_1'|_{y=y_1+0}$$

we will have

$$
\begin{aligned}
m_1' &= \mu\varphi(z), \qquad \mu y = \mu y_0 + \frac{2T_1(z_0, z)}{b_6}, \\
\mu y_1 &= \mu y_0 + \frac{2T_1(z_0, z_1)}{b_6}
\end{aligned} \tag{5.4}
$$

at $y_0 \leq y \leq y_1$ and

$$m_1' = \mu\psi(\zeta), \qquad \mu y = \mu y_1 + \frac{2T_2(\zeta_0, \zeta)}{b_3} \tag{5.5}$$

at $y_1 \le y \le 2$. Here we use the notations

$$\varphi(z) = \left[\left(\frac{m'_{10}}{\mu}\right)^2 + \frac{z-z_0}{b_6}\chi_1(z)\right]^{\frac{1}{2}}, \psi(\zeta) = \left[\varphi^2(z_1) + \frac{\zeta-\zeta_0}{b_3}\chi_2(\zeta)\right]^{\frac{1}{2}},$$

$$T_1(z_0, z) = \int_{z_0}^{z} \frac{z\,\mathrm{d}z}{\varphi(z)}, \qquad T_2(\xi_0, \xi) = \int_{\xi_0}^{\xi} \frac{\xi\,\mathrm{d}\xi}{\psi(\xi)},$$

$$\chi_i(z) = \tfrac{2}{3}[3q_i^0(z+z_0) - 2(z^2 + zz_0 + z_0^2)], \qquad (i = 1, 2),$$

$$z = (b_5 + b_6 m_1)^{\frac{1}{2}}, \qquad z_0 = (b_5 + b_6 m_{10})^{\frac{1}{2}}, \qquad z_1 = (b_5 + b_6 m_1^c)^{\frac{1}{2}},$$

$$\xi = (b_2 + b_3 m_1)^{\frac{1}{2}}, \qquad \xi_0 = (b_2 + b_3 m_1^c)^{\frac{1}{2}}, \qquad \xi_1 = (b_2 + b_3 m_1^b)^{\frac{1}{2}},$$

$$q_1^0 = p_1 - t_2^* - b_4, \qquad q_2^0 = p_1 - t_2^* - b_1,$$

$$m'_{10} = m'_1|_{y=y_0}, \qquad m_{10} = m_1|_{y=y_0}.$$

For the waffle-type shells $m_{10} = m'_{10} = 0$, $y_0 = 0$. Just as in [8] one can show that at the clamped end of the shell a plastic hinge is formed and the corresponding boundary condition has the form

$$m_1|_{y=2} = m_1^b. \tag{5.6}$$

Taking into consideration this condition from (5.4) for all the above cases we obtain the dependence of $\mu$ on $p_1$ in the form

$$\mu = \frac{T_1(z_0, z_1)}{b_6} + \frac{T_2(\xi_0, \xi_1)}{b_3}. \tag{5.7}$$

At $\omega_1 = \omega_2 = 1$, $h_1 = h_2$, $\lambda_1 = \lambda_2$ we obtain the dependence for a bimetallic shell and at $\omega_1 = \omega_2 = h_1 = h_2 = 0$ for a uniform smooth shell.

The limiting relations for cylindrical shells strengthened by longitudinal ribs are obtained from relations (3.1)–(3.15) assuming $\omega_2 = \lambda_2 = 0$, $v_2^- = v_1^+ = 0$, $v_1^- = 2\omega_1\lambda_1$. In this case surfaces (3.4), (3.5) are missing while (3.6) and (3.7) degenerate. The remaining limiting correlations in the general case determine in the plane $t_1 = 0$ the limiting curve plotted in Fig. 6. In case of $t_1^* \le 1$ and the internal position of the strengthening ribs ($\alpha = 1$) the parabolas $bc$ and $cd$ are determined, respectively, by equations (3.11) and (3.10) at $t_1 = 0$, $\omega_2 = \lambda_2 = 0$, $v_2^- = v_1^+ = 0$, $v_1^- = 2\omega_1\lambda_1$. If the strengthening ribs are placed externally ($\alpha = -1$) the parabola $bc$ is analogously determined from equation (3.11) and parabola $cd$ degenerates into a point. The abscissa $m_1^b$, is determined from equation (3.14) at $\alpha = 1$, $t_1 = 0$.

In case of $t_1^* \ge 1$ and the internal position of the strengthening ribs ($\alpha = 1$) the parabola $bc$ is again determined by equation (3.11) and the parabola $cd$ degenerates into a point. If the strengthening ribs are placed externally the curve has the form of the dotted rectangle *abkde* in Fig. 6. The abscissa $m_1^b$ is determined from equation (3.15) at $t_1 = 0$.

The straight line *de* in all cases of the shell strengthened by longitudinal ribs is determined by the equation

$$t_2 = 1. \tag{5.8}$$

Writing the equation for parabolas $bc$ and $cd$ again in the form (5.2) in the regions $y_0 \le y \le y_1$ and $y_1 \le y \le 2$ we will obtain the solutions (5.3)–(5.4).

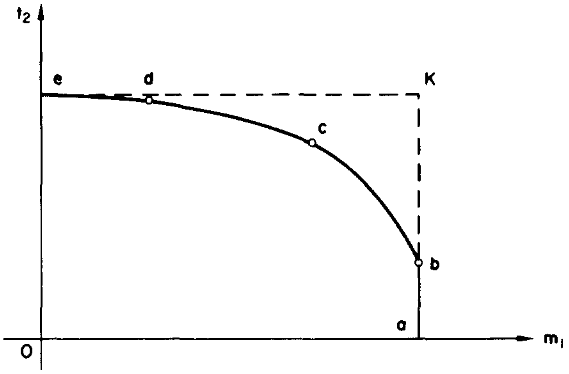


FIG. 6.

Substituting the value of (5.8) into (5.1) and integrating the resulting equation at the boundary conditions (5.3) and using the continuity conditions:

$$m_1|_{y=y_0-0} = m_1|_{y=y_0+0} = m_1^d, m_1'|_{y=y_0-0} = m_1'|_{y=y_0+0}$$

we will obtain the solution in the region $0 \leq y \leq y_0$

$$\begin{aligned} m_1' &= \mu^2(p_1-1)y, \qquad m_1 = \mu^2(p_1-1)y^2/2, \\ \mu y_0 &= \left(\frac{2m_1^d}{p_1-1}\right)^{\frac{1}{2}}, \qquad m_{10} = m_1^d, \qquad m_{10}' = \mu[2m_1^d(p_1-1)]^{\frac{1}{2}}. \end{aligned} \tag{5.9}$$

From (5.5), using the boundary condition (5.6), we obtain the dependence (5.7) of $p_1$ on $\mu$ for a shell strengthened with internally placed ribs.

To obtain a similar dependence for a shell strengthened externally ($\alpha = -1$) at $t_1^* \leq 1$ or for a shell strengthened internally but at $t_1^* \geq 1$, we must take the above changes into account and substitute into (5.7) $m_1^b = m_1^c$. As a result we will have

$$\mu = \frac{T_1(z_0, z_1)}{b_6} \qquad (\text{at } m_1^c = m_1^b). \tag{5.10}$$

But if a shell is strengthened externally ($\alpha = -1$) by ribs whose rigidity $t_1^* \geq 1$, then by substituting into (5.9)

$$m_1^d = m_1^k = m_1^b, y_0 = 2$$

we will obtain the dependence

$$p_1 = 1 + \frac{m_1^b}{2\mu^2}. \tag{5.11}$$

Using the expressions (5.2), (5.4), (5.5), (5.8) and (5.9) the flow law (2.6) and relations of the type (4.2) for cylindrical shells one can readily see that all the necessary kinematical conditions are satisfied. Hence the obtained solution is exact.

To obtain a solution for shells strengthened by both longitudinal and transverse ribs it is sufficient to substitute $p_1' = p_1 - t_2^*$ for $p_1$ in all formulae (5.2)–(5.11).

Note that all solutions obtained in this section are also valid in case of $0 \leq t_1 \leq 1 + t_1^*$, if the coefficients $b_1, b_2, b_3, b_4, b_5, b_6$ in (5.2) are determined accordingly from the relations (3.1)–(3.15).

The solutions obtained permit an estimation of the efficiency of using one or other type of strengthening. To this end, alongside with the given strengthened shell, consider a shell of the same load-carrying capacity but smooth and uniform with length $2L$, radius $R$ and thickness $2H_0$ made of a material with yield stress $\sigma_0^0$.

As their load-carrying capacities are equal, we take the expression for $\mu, p_1$ in (5.1) and obtain the following condition of equal strength [9]

$$\frac{p_1}{\mu^2} = \frac{\sigma_0^0 p_1^0}{\sigma_0 \mu_0^2} = c.$$

As $\sigma_0 p_1 H = \sigma_0^0 p_1^0 H_0$, the relative weight of waffle-type shells (with respect to that of the equivalent smooth one) is obtained from the formula

$$b = \frac{\sigma_0^0 p_1^0 \rho}{\sigma_0 p_1 \rho_0}\left\{1 + \frac{\rho[t_2^* + (1-\omega_2)t_1^*]}{\lambda \rho_1^0}\right\}. \tag{5.12}$$

Here $\rho, \rho_1^0$ are the specific weights of the material of the facing and that of the strengthening layer, $\rho_0$ the specific weight of the equivalent shell, $\lambda = \lambda_1 = \lambda_2$ is the non-dimensional yield stress of the strengthening layer.

For the ribbed shell we have [16]

$$b = \frac{\sigma_0^0 p_1^0 \rho}{\sigma_0 p_1 \rho_0}\left(1 + \frac{\rho t_1^*}{\lambda_1 \rho_1} + \frac{\rho t_2^*}{\lambda_2 \rho_2}\right) \tag{5.13}$$

where $\rho_1, \rho_2$ are the specific weights of the rib materials. To obtain the values of $p_1$ contained in (5.12) and (5.13) we must determine the coordinates of the intersection points of the load-carrying capacity curves (the plot for $p_1 vs \mu$) for the strengthened shell with the parabola

$$p_1 = c\mu^2. \tag{5.14}$$

Similarly, to determine $p_1^0$ it is necessary to find the coordinates of the intersection, points of the load-carrying capacity curve for the smooth shell with the parabola

$$p_1 = \frac{c\sigma_0}{\sigma_0^0}\mu^2.$$

Substituting the values of $p_1$ and $p_1^0$ obtained at the given values of $t_1^*, t_2^*$ and $h_1, h_2$ into (5.12) and (5.13) we will determine the relative weight. For the cylindrical waffle-type shells with clamped edges a similar calculation has been done in [9]. The efficiency of various types of strengthening for some kinds of shell loading and edge fastening has been estimated in [16].

In Fig. 7, for illustration, there are given the diagrams of the limit load dependences $p_1 vs \mu$ for the waffle-type shells at $h_1 = h_2 = 4$, $\lambda_1 = \lambda_2 = 1$ calculated by formulas (5.7). Curve 1 corresponds to a smooth shell, ($t_1^* = t_2^* = 0$), curve 2 corresponds to the values $t_1^* = t_2^* = 0{\cdot}25$, while curve 3 corresponds to the values $t_1^* = 0{\cdot}5$, $t_2^* = 1$, and curve 4 corresponds to the values $t_1^* = 0{\cdot}8$, $t_2^* = 0{\cdot}4$. Here are also given the parabolas of equistrength (5.14) for different values $c$.

Note that for $t_1^* = t_2^*$, $h_1 = h_2$, $\lambda_1 = \lambda_2 = 1$ the load carrying capacity of shells does not depend on the position of the stiffening layer. For $t_1^* \neq t_2^*$ there has been discovered a weak dependence of the load carrying capacity of shells on the position of the stiffening layer. For the parameter values in question the difference is 0·5–1·5 per cent. It should be noted that for $t_1^* < t_2^*$ the load carrying capacity of the shell is higher at the exterior position of the stiffening layer, being lower at $t_1^* > t_2^*$. Similar results have been obtained in paper [9] for the edge-clamped shell, the difference being 4–10 per cent.

In Fig. 8 the dependences of the relative weight $b$ on the quantity $c$ for the parameter values in question are given. Analysis of these dependences permits one to draw the following conclusion:

(1) The relative weight for the stiffened shells decreases with decreasing length,

(2) Increase of the specific volume of the circumferential ribs in comparison with that of the longitudinal ribs is unreasonable from the standpoint of weight.

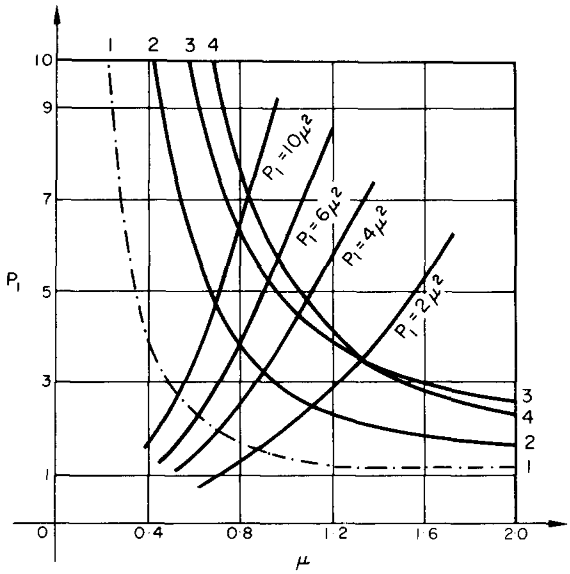


FIG. 7.

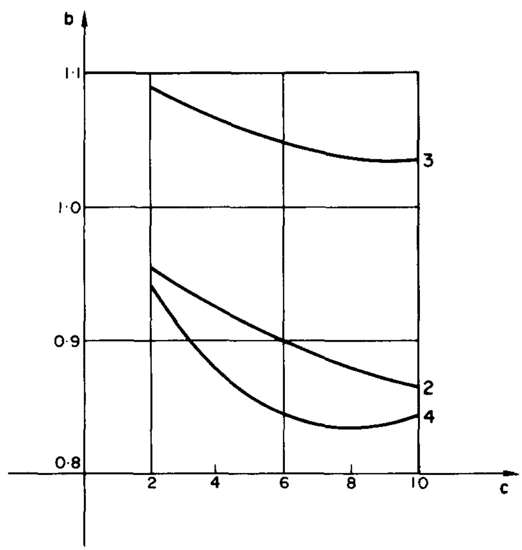


FIG. 8.

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**Абстракт**—Получены точные предельные соотношения для подкрепленных осесиммтеричных оболочек. При этом в зависимости от учета или неучета влияния узлов пересечения подкрепляющих элементов оболочки подразделяется на подкрепленные и вафлеобразные. В частных случаях найден ные соотношения сводятся к предельным соотношениям для оболочек, подкрепленных ребрами одного направления, для биметаллических и однородных изотропных оболочек. Предлагаются приближенные предельные соотношения. Приводится краткий обзор результатов анализа других типов подкрепленных оболочек. Обсуждается вопрос линеаризации предельных соотношений за счет моделирования рассматриваемых оболочек слоистыми моделями. В качестве примера использования полученных предельных соотношений решена задача о несущей способности консольной подкрепленной цилиндрической оболочки, нагруженной внутренним гидростатическим давлением Получены формулы, позволяющие оценить эффективность использования подкрепляющего набора.